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2003 J. Phys. A: Math. Gen. 36 L463

(http://iopscience.iop.org/0305-4470/36/32/101)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) L463–L466 PII: S0305-4470(03)63796-0

LETTER TO THE EDITOR

Uniqueness of a negative mode about a bounce solution

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Received 21 May 2003
Published 29 July 2003
Online at stacks.iop.org/JPhysA/36/L463

Abstract

We consider the uniqueness problem of a negative eigenvalue in the spectrum of small fluctuations about a bounce solution in a multidimensional case. Our approach is based on the concept of conjugate points from Morse theory and is a natural generalization of the nodal theorem approach usually used in the one-dimensional case. We show that the bounce solution has exactly one conjugate point at $\tau = 0$ with multiplicity one.

PACS numbers: 03.65.Sq, 03.65.Xp, 03.65.−w

In the leading semiclassical approximation tunnelling transitions are associated with classical solutions of Euclidean equations of motion. A special type of such solution, time-reversal invariant solution, that approaches the local minimum $\vec{q}_f$ of a potential $V(\vec{q})$ at infinity referred to as a bounce, dominates the WKB transition rate from the bottom of the potential well. In the quantum version of the theory, the classically stable equilibrium state $\vec{q}_f$ becomes unstable through barrier penetration. It is a false vacuum state. Callan and Coleman [1] approached the problem of false vacuum decay by evaluating the Euclidean (imaginary time) functional integral at the bounce solution in the semiclassical (small-$\hbar$) limit. They found that in this way the negative eigenvalue of the second variational derivative of the action at the bounce makes an imaginary energy shift and may be interpreted as a decay rate. Since this interpretation relies heavily upon the existence of a unique negative mode, it is crucial to show that the second variational derivative of action at the bounce has one and only one negative eigenvalue. The negative mode problem was already considered by Coleman [2]. We reconsider this problem by using the concept of conjugate points. The discussion presented here has a potential application in quantum field theory viewing a field $\phi(\vec{x})$ as a collection of mechanical variables q^i $(i = 1, \ldots, N)$ for N degrees of freedom, in the limit N becomes continuously infinite: $\vec{q} \to \phi(\vec{x}), i \to \vec{x}$. However, to take this limit on a quite sound mathematical ground requires the use of a rigorous functional analysis. Throughout this letter we will restrict ourselves to the consideration of a multidimensional case. We shall assume that the initial point of tunnelling $\vec{q}_{\text{in}}$ is taken arbitrarily. In this case tunnelling is described by the solution of the imaginary-time equations of motion which begin at some position $\vec{q}_{\text{es}} \neq \vec{q}_{\text{in}}$ at rest and come to rest at time $T/2$ at $\vec{q}_{\text{in}}$. For the sake of convenience we take $V(\vec{q}_{\text{in}}) = 0$. Thus, the solution we are interested in is defined from this zero-energy solution by the time reflection, $\vec{q}_b(\tau) = \vec{q}_b(-\tau)$. (The suffix b denotes a bounce-like solution, in a particular case

$T = \infty$ we arrive at the bounce.) By its definition, this is an even, zero-energy stationary point of action,

$$S_E[\vec{q}] = \int_{-T/2}^{T/2} \mathrm{d}\tau \left(\frac{m_{ik}(\vec{q})\dot{q}^i\dot{q}^k}{2} + V(\vec{q}) \right) \tag{1}$$

with the boundary conditions $\vec{q}_b(\pm T/2) = \vec{q}_{\text{in}}$. Here $m_{ik}(\vec{q})$ is some positive-definite symmetric matrix and the summation convention over repeated indices is used. The matrix $m_{ik}(\vec{q})$ defines the metric in a configuration space, $(\mathrm{d}\vec{q})^2 = m_{ik}\mathrm{d}q^i\mathrm{d}q^k = \mathrm{d}q_k\mathrm{d}q^k$. The corresponding imaginary-time equations of motion take the form

$$\frac{\delta^2 q^i}{\delta\tau^2} - \frac{\partial V}{\partial q_i} = 0 \tag{2}$$

where $\frac{\delta}{\delta\sigma} = \dot{q}^i\nabla_i = \dot{q}_i\nabla^i$ is a covariant derivative along the vector $\dot{q}^i$ and ∇_i is the covariant derivative with respect to q^i compatible with metric m_{ik}. According to the formalism developed by Banks *et al* [3], in the leading semiclassical approximation the tunnelling probability is dominated by the solution that minimizes the Jacobi-type action. Now, in order to define the corresponding Jacobi-type action, we introduce a parameter σ along the path, $\vec{q}(\sigma)$, that increases monotonically from $-T/2$ at initial point $\vec{q}_{\text{in}}$ to 0 at the escape one $\vec{q}_{\text{es}}$. Denoting $\dot{\vec{q}} \equiv \frac{\mathrm{d}\vec{q}(\sigma)}{\mathrm{d}\sigma}$, the action is given by the functional

$$J_E[\vec{q}] = \int_{-T/2}^{0} \mathrm{d}\sigma \sqrt{2V(\vec{q})m_{ik}(\vec{q})\dot{q}^i\dot{q}^k} \tag{3}$$

over trajectories $\vec{q}(\sigma)$ connecting two boundary points on different sides of the barrier, $\vec{q}(-T/2) = \vec{q}_{\text{in}}$ and $\vec{q}(0) = \vec{q}_{\text{es}}$. In general, the barrier penetration path has to be at least a local minimum of this action. This is crucial for our further discussion. An extremum of this action gives a classical path in the configuration space of the system, but says nothing about its motion in imaginary time. To determine the system's evolution in imaginary time requires the use of a supplementary condition,

$$\frac{m_{ik}(\vec{q})}{2}\frac{\mathrm{d}q^i}{\mathrm{d}\tau}\frac{\mathrm{d}q^k}{\mathrm{d}\tau} - V(\vec{q}) = 0. \tag{4}$$

The variational principles of mechanics are widely discussed in [4]. This extra relation is just the Euclidean energy condition and once the configuration path $\vec{q}(\sigma)$ is known, it can be integrated to get the imaginary-time parametrization $\sigma(\tau)$. The equation of motion following from the Jacobi-type action (3) and the supplementary condition (4) is equivalent to the imaginary-time equation (2) with the Euclidean energy, a first integral of Euclidean equation (2), fixed to the value zero. This is shown explicitly by varying (3), which yields the equation of motion

$$\left(\frac{\delta^2 q^i}{\delta\sigma^2} - \frac{\dot{\vec{q}}^2}{2V(\vec{q})}\nabla^i V \right) \Pi_i^k = 0 \tag{5}$$

where $\Pi_i^k = \delta_i^k - \dot{q}^k\dot{q}_i/\dot{\vec{q}}^2$ is the projection operator onto the subspace of configuration space that is orthogonal to the configuration space path. If by using equation (4) we parametrize a configuration space path $\vec{q}(\sigma)$ with parameter τ, equation (5) just becomes

$$\left(\frac{\delta^2 q^i}{\delta\tau^2} - \nabla^i V \right) \Pi_i^k = 0. \tag{6}$$

Therefore, the equation of motion (5) obtained from the Jacobi-type action (3), supplemented by (4), is equivalent to imaginary-time equation of motion (2) with fixed zero Euclidean energy. The parallel projection of equation (2) to the configuration path follows from equation (4), by differentiating with respect to τ,

$$\left(\frac{\delta^2 q^i}{\delta\tau^2} - \nabla^i V\right)\frac{\mathrm{d}q_i}{\mathrm{d}\tau} = 0. \tag{7}$$

Note that if we multiply equation (7) by the tangential vector $\frac{\mathrm{d}q^k}{\mathrm{d}\tau}\Big/m_{nl}(\vec{q})\frac{\mathrm{d}q^n}{\mathrm{d}\tau}\frac{\mathrm{d}q^l}{\mathrm{d}\tau}$ and add to equation (6) we get equation (2). As an essential point for our discussion we want to emphasize that the Jacobi-type action (3) is invariant under the reparametrizations of the configuration space path that preserve the end point values of the parameter. That is, (3) is invariant under the replacements $\sigma \to f(\sigma)$ and $q^i(\sigma) \to \bar{q}^i(f(\sigma))$ with $f(-T/2) = -T/2$ and $f(0) = 0$. Their infinitesimal form is $\sigma \to \sigma + \epsilon(\sigma)$ and $q^i \to q^i + \epsilon\dot{q}^i$ where $\epsilon(-T/2) = \epsilon(0) = 0$. Due to Noether's second theorem, there is a corresponding gauge identity of the form (7). Now it is obvious that the proper fluctuations for Jacobi-type action are transverse ones, while the longitudinal ones reproduce a gauge transformation. Now let us recall some definitions and statement from Morse theory about the conjugate points. The Morse index for a given trajectory is defined as a number of negative eigenvalues of a second variation of action evaluated at this trajectory with the zero boundary conditions at the end points. On the other hand, due to the Morse theory the number of negative eigenvalues may be evaluated by counting the conjugate points with their multiplicities [5]. The point c is conjugate to a with multiplicity m for the differential operator A if the two-point boundary value problem $Au = 0, u(a) = u(c) = 0$ has the nontrivial m linearly independent solutions. The second variation of the action (3) in imaginary-time parametrization gives the following two-point boundary value problem:

$$\left(-\frac{\delta^2\phi^i}{\delta\tau^2} - R^i_{jkl}\dot{q}^j_b\phi^k\dot{q}^l_b + (\nabla^i\nabla_j V)\phi^j\right)\Pi^k_i(b) = 0 \qquad \vec{\phi}(-T/2) = \vec{\phi}(0) = \mathbf{0} \tag{8}$$

where $\Pi^k_i(b)$ denotes orthogonal projection onto the bounce-like solution. We are tacitly assuming here and below that the terms R^i_{jkl} (the Riemann curvature tensor of a metric m_{ik}), $\nabla^i\nabla_j V$ are evaluated at $\vec{q}_b$ and $\frac{\delta}{\delta\tau} = \frac{\mathrm{d}q^i_b}{\mathrm{d}\tau}\nabla_i$. The barrier penetration path is at least a local minimum of action (3), i.e. all small transverse (proper) fluctuations about this path increase this action. It means that $\vec{\phi}$ must not contain a transverse part. On the other hand, since the barrier penetration path is at least a local minimum of action (3) the second variation at this path must be a positive semidefinite operator; in such a case due to Morse theory the $\vec{\phi}$ must be free of conjugate points. Considering the following two-point boundary value problem,

$$-\frac{\delta^2\psi^i}{\delta\tau^2} - R^i_{jkl}\dot{q}^j_b\psi^k\dot{q}^l_b + (\nabla^i\nabla_j V)\psi^j = 0 \qquad \vec{\psi}(-T/2) = \vec{\psi}(0) = \mathbf{0} \tag{9}$$

one concludes that the solutions $\vec{\psi}$ automatically satisfy the two-point boundary value problem (8). It means that $\vec{\psi}$ also must not contain a transverse part and the conjugate points. Taking a general longitudinal ansatz satisfying the corresponding boundary conditions $\vec{\psi} = \lambda(\tau)\frac{\mathrm{d}\vec{q}_b}{\mathrm{d}\tau}$, from equation (9) we get the equation $\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{\mathrm{d}\lambda(\tau)}{\mathrm{d}\tau}\left(\frac{\mathrm{d}\vec{q}_b}{\mathrm{d}\tau}\right)^2\right] = 0$. It gives $\lambda = \mathrm{const}$ and correspondingly $\vec{\psi} \sim \frac{\mathrm{d}\vec{q}_b}{\mathrm{d}\tau}$. Combining these, we obtain the following proposition.

Proposition 1. *All solutions of the two-point boundary value problem (9) are linearly dependent and free of conjugate points.*

Now we turn our attention to a consideration of the second variation of equation (1) evaluated at the bounce-like solution,

$$-\frac{\delta^2\chi^i}{\delta\tau^2} - R^i_{jkl}\dot{q}^j_b\chi^k\dot{q}^l_b + (\nabla^i\nabla_j V)\chi^j = 0 \qquad \vec{\chi}(\pm T/2) = \mathbf{0}. \tag{10}$$

Since the solution $\vec{q}_b(\tau)$ is an even function of τ, the second variational derivative in equation (10) commutes with the operator $\hat{T}$, $\hat{T} : \vec{\chi}(\tau) = \vec{\chi}(-\tau)$. Therefore, the function χ is to be either even or odd with respect to τ. Due to this symmetry the presence of the conjugate point $\tau_{\mathrm{conj}} > 0$ implies the presence of the second conjugate point $-\tau_{\mathrm{conj}}$, but due to proposition 1 this is not the case. Nevertheless, the solution $\vec{\chi} \sim \frac{\mathrm{d}\vec{q}_b}{\mathrm{d}\tau}$ has a conjugate point $\tau = 0$. The multiplicity of this conjugate point equals one. Namely, from proposition 1 we know that all solutions of two-point boundary value problem (9) are linearly dependent. Thus, we arrive at the following result.

Proposition 2. *The two-point boundary value problem (10) has exactly one conjugate point* $\tau = 0$ *with multiplicity one.*

Due to Morse theory this means the presence of a unique negative mode in the spectrum of small fluctuations about the bounce solution.

Acknowledgments

I am very grateful to my advisor in Tbilisi State University, Professor A Khelashvili, for helpful conversations and to Professor G Lavrelashvili for useful discussions.

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